

# Eisenstein Series on Arithmetic Quotients of Rank 2 Kac–Moody Groups Over Finite Fields

Abid Ali  
*Rutgers University*

Lie Group and Quantum Mathematics Seminar

Joint work with Lisa Carbone, Kyu-Hwan Lee, Dongwen Liu

**Lisa Carbone, Yi-Zhi Huang, Jim Lepowsky and  
Siddhartha Sahi**

*Thank you*

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- ▶ There is a natural *adjacency operator* on  $X$ , which induces an operator  $T_\Gamma$  on the quotient graph  $\Gamma/X$ .
- ▶ Eisenstein series are eigenfunctions for  $T_\Gamma$ , whose eigenvalues represent the continuous spectrum of  $T_\Gamma$ .

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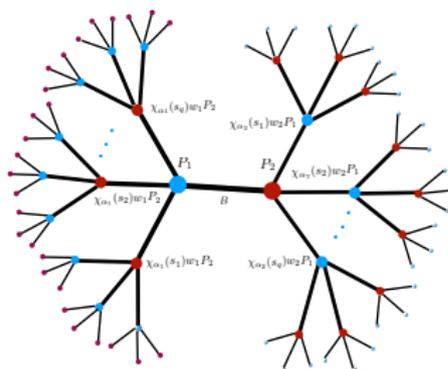
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- ▶ The associated Bruhat Tits building  $X$  is a tree.



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- ▶ We discuss its constant term, its convergence and meromorphic continuation.

# Related Work

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- ▶ Beyond affine type, Carbone et. al studied the Eisenstein series of hyperbolic Kac-Moody groups.
- ▶ Our work is a locally compact analog since the tree is locally finite. This gives us Haar measure that allows us to carry out integration on the Kac-Moody group.

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- ▶ We denote by  $\mathfrak{g}$  the Kac-Moody algebra associated with  $A$ .
- ▶ As a Lie algebra  $\mathfrak{g}$  is generated by Chevalley generators  $\{e_i, f_i \mid i = 1, 2\}$  and  $\mathfrak{h}$ .

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- ▶ Once  $k$  and  $\lambda$  are fixed, we drop it from the notation and just write  $G_0$ .

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- ▶ The subgroups  $U_0^+$  and  $U_0^-$  are the analogs of upper and lower triangular subgroups, respectively.

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- ▶ The *standard parabolic subgroups* are given as

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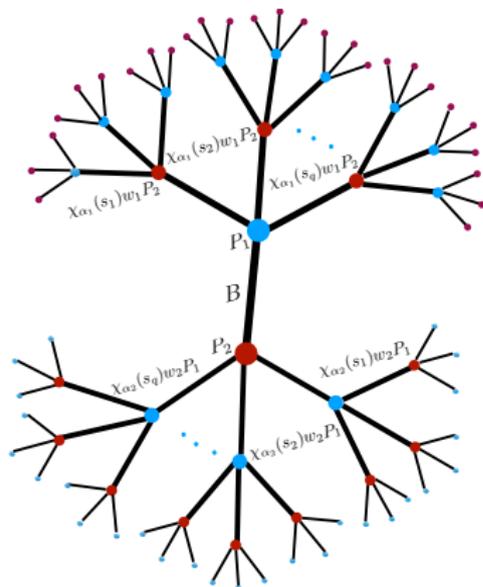
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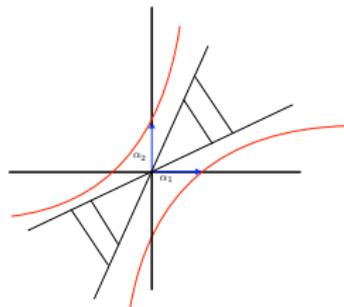
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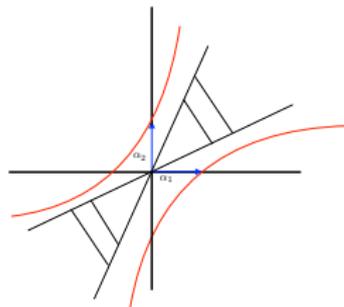
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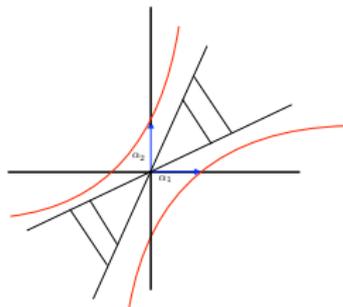


- ▶ Let  $\mathcal{U}_i$  be the closure of the group generated by  $U_\alpha$  with  $\alpha \in \Delta_i^{re}$  and  $K = P_i$ , for  $i = 1, 2$ , Carbone and Garland show that  $G = \mathcal{U}_1AK = \mathcal{U}_2AK$ .

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$$\Delta_i^{re} := \{-\alpha_{3-i}, -w_{3-i}\alpha_i, \dots\} \cup \{\alpha_i, w_i\alpha_{3-i}, \dots\}.$$

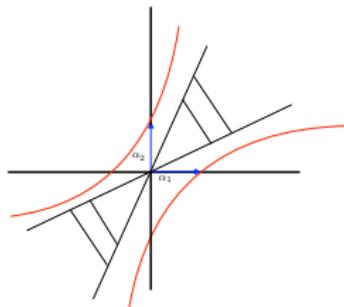


- ▶ Let  $\mathcal{U}_i$  be the closure of the group generated by  $U_\alpha$  with  $\alpha \in \Delta_i^{re}$  and  $K = P_i$ , for  $i = 1, 2$ , Carbone and Garland show that  $G = \mathcal{U}_1AK = \mathcal{U}_2AK$ .
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- ▶ Using the Iwasawa decomposition, we write  $G = I_1 \sqcup I_2$ , where
- ▶ for  $i = 1, 2$

$$I_i := \left( \bigcup_{x \in \mathcal{U}_i} \bigcup_{n \in \mathbb{Z}_{\geq 0}} x(w_i w_{3-i})^n P_1 \right) \cup \left( \bigcup_{x \in \mathcal{U}_i} \bigcup_{n \in \mathbb{Z}_{\geq 0}} x(w_i w_{3-i})^n P_2 \right).$$

# Iwasawa Decomposition and Vertices

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Main Steps in Proof:

- For  $w \in W$ , set  $S_w^\pm = \Delta_\pm \cap w(\Delta_\mp)$  and define the following finitely generated subgroups  $U_{0,w}^\pm = \langle U_\alpha \mid \alpha \in S_w^\pm \rangle$ .
- We let  $U_0^{\pm,w}$  be the subgroup generated by  $U_\alpha$  for  $\alpha \in \Delta_\pm^{re} \setminus S_w^\pm$ .

## Lemma

*If  $\lambda$  is regular and  $w \in W$ , then  $U_0 = U_0^w U_{0,w} = U_{0,w} U_0^w$ .*

## Corollary

*For  $w \in W$  and  $\lambda$  regular, we have  $U = U^{+,w} U_w = U_w U^{+,w}$ .*

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- ▶ Let the elements of the sets  $\Omega_v^1$  and  $\Omega_v^q$  correspond to the Iwasawa cell with  $A$ -component lengths  $k^1(i)$  and  $k_m^q(i)$ , respectively, where  $m = 1, 2, \dots, q$ .

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The following local picture gives a graphical interpretation of the above proposition.

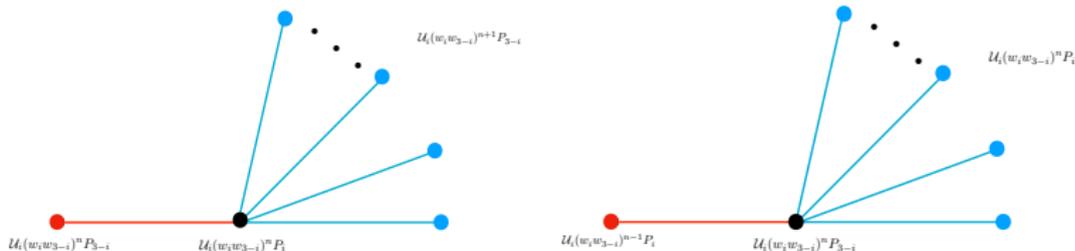


Figure: Local picture of adjacent vertices

# Character and Adjacency Operator

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- For  $i = 1, 2$ ,  $s \in \mathbb{C}^*$  and  $g \in \mathcal{U}_i(w_i w_{3-i})^{n(i)} P_j$  we set

$$\Psi_{i,s}(g) = \begin{cases} q^{-s2n(i)} & \text{if } j = i \\ q^{-s(2n(i)-1)} & \text{if } j = 3 - i. \end{cases}$$

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## Theorem

*The character  $\Psi_{i,s}$  is an eigenfunction for the adjacency operator  $T$  with an eigenvalue  $q^{1-s} + q^s$ .*

# Eisenstein series

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- ▶ Define the Eisenstein series  $E_{i,s}$  on  $\Gamma \backslash X = \sqcup_{i=1}^2 \Gamma \backslash G / P_i$ ,

$$E_{i,s}(gP_j) = \sum_{\gamma \in (\Gamma \cap \mathcal{U}_i) \backslash \Gamma} \Psi_{i,s}(\gamma g P_j), \quad j = 1, 2.$$

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## Proposition

*Let  $\Gamma$  be a lattice subgroup of  $G$ ,  $\gamma = u_\gamma a_\gamma k_\gamma \in \Gamma$  with  $u_\gamma \in \mathcal{U}_i$ ,  $a_\gamma = (w_i w_{3-i})^{n_i(\gamma)} \in A$ ,  $i = 1, 2$ ,  $k_\gamma \in K$ . Then  $n_i(\gamma)$  is bounded below, that is, there exists  $n_0 \in \mathbb{Z}$  such that  $n_i(\gamma) \geq n_0$  for all  $\gamma \in \Gamma$ .*

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### Proposition

We have

$$C_{\mathcal{U}_1}(E_{1,s}) = \Psi_{1,s} + c(s)\Psi_{1,1-s},$$

where  $c(s) \in \mathbb{C}$  is a constant depending on  $s$ .

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- ▶ For this we define the Seigel sets and show that they have finite self intersection.
- ▶ Consequently, a certain truncation operator lives on a finite graph and hence it is compact.
- ▶ Using the compact operator criterion with Bernstein method, we obtain the the required continuation.

# Final Remarks

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- ▶ As stated earlier, in our previous work we have defined congruence subgroups of  $\Gamma$ , constructed their graph of groups and have shown that those are lattice subgroups.
- ▶ The future direction for the current work is to extend our constructions for these congruence subgroups.

Thank You